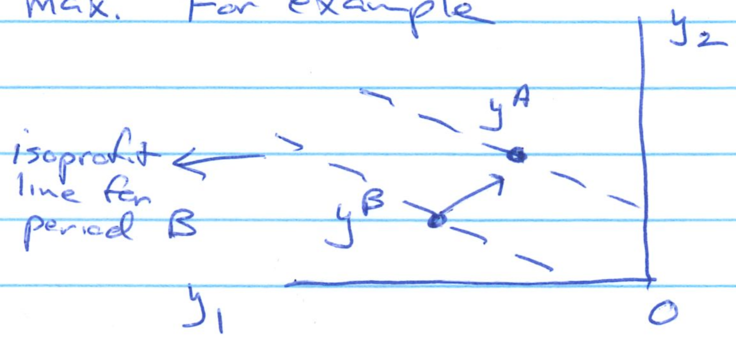
Econ 802
Midterm 1 Answers

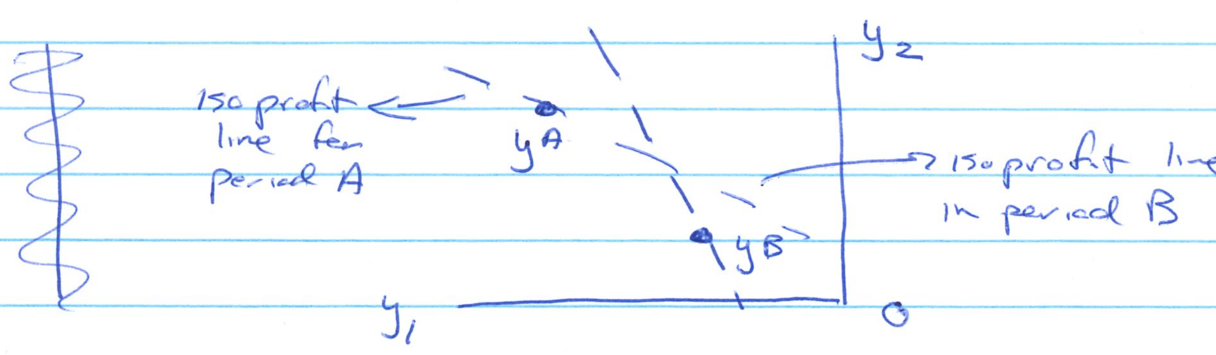
Greg Dow October 2019

1 (a) If one production plan is above and to the right of the other, this can never be consistent with profit max. For example



No matter what the prices p^B may be, the isoprofit line through y^B must be below y^A, so in period B the firm would get more profit by choosing y^A.

The same is true if one plan is directly above the other, or directly to the right.
However, if one plan is above and to the left of the other, there are always prices for which the firm's behavior is consistent with the Weak Axiom of Profit Max. For example:



(b) We must have $p_1^A y_1^A + p_2^A y_2^A \geq p_1^A y_1^B + p_2^A y_2^B$

$\Rightarrow p_1^A(y_1^A - y_1^B) \geq p_2^A(y_2^B - y_2^A)$

$\Rightarrow \dfrac{p_1^A}{p_2^A} \leq \dfrac{y_2^B - y_2^A}{y_1^A - y_1^B}$ (assuming $y_1^A < y_1^B$ as shown in the graph, ie in period A the firm uses more of the input in absolute value so y_1^B is less negative than y_1^A)

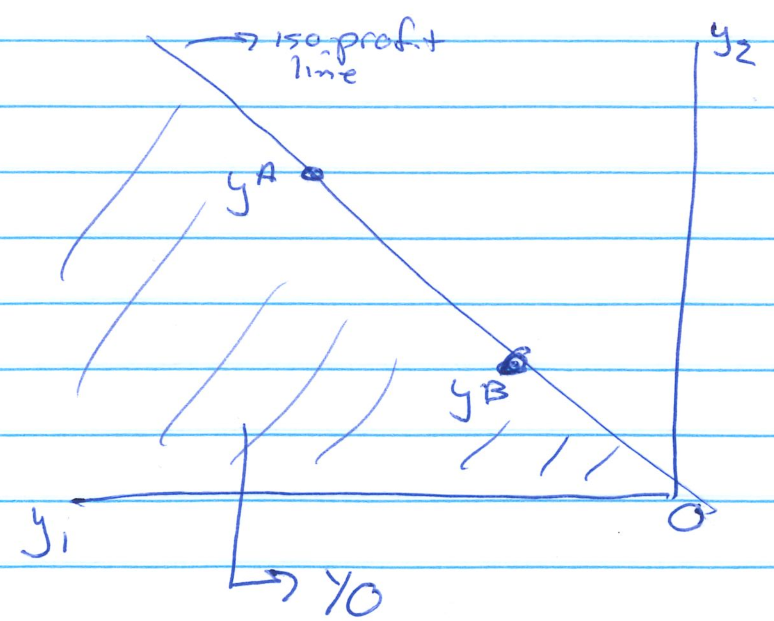
Similarly, we must have $p_1^B y_1^B + p_2^B y_2^B \geq p_1^B y_1^A + p_2^B y_2^A$

$\Rightarrow \quad p_1^B (y_1^B - y_1^A) \geq p_2^B (y_2^A - y_2^B)$

$\Rightarrow \quad \dfrac{p_1^B}{p_2^B} \geq \dfrac{y_2^A - y_2^B}{y_1^B - y_1^A}$ again assuming $y_1^A \leq y_1^B$ as in the graph.

or to sum up, $\dfrac{p_1^B}{p_2^B} \geq \dfrac{y_2^A - y_2^B}{y_1^B - y_1^A} \geq \dfrac{p_1^A}{p_2^A}$

1(c) If the price vectors are identical in periods A and B, we get the same isoprofit lines, with slopes $= -\dfrac{p_1}{p_2}$. y^A cannot be on a higher isoprofit line than y^B because this would contradict profit max in period B and vice versa. So y^A and y^B must be on the same isoprofit line which means the slope $-\dfrac{p_1}{p_2}$ must be equal to the slope of the line passing through y^A and y^B (using the results from part (b), this gives $\dfrac{p_1}{p_2} = \dfrac{y_2^A - y_2^B}{y_1^B - y_1^A}$).



YO includes all points on this isoprofit line or below it (this set is monotonic, convex, and closed). YO does not include any point on a higher isoprofit line. Note that YO may or may not include the origin, depending on where y^A and y^B are located.

2 (a) If the Hessian is negative definite everywhere, we can use the first order condition method.

FOC: $p \dfrac{\partial f(x^*)}{\partial x} = w$. Let $x(p,w)$ be the unconditional input demands.

Can use implicit function theorem: $p \dfrac{\partial f[x(p,w)]}{\partial x} \equiv w$

Differentiate with respect to w:

$$p \frac{\partial^2 f[x(p,w)]}{\partial x^2} \frac{\partial x(p,w)}{\partial w} = I$$

Because the Hessian is neg def, it is non-singular so

$$\frac{\partial x(p,w)}{\partial w} = \left[p \frac{\partial^2 f[x(p,w)]}{\partial x^2} \right]^{-1}$$

The inverse of a neg def. matrix is also neg def, so diagonal elements of $\dfrac{\partial x(p,w)}{\partial w}$ are strictly negative.

Thus $\dfrac{\partial x_i(p,w)}{\partial w_i} < 0$ for all $i = 1 \ldots n$

(unconditional input demand curves slope down)

(b) From Hotelling's Lemma, $\dfrac{\partial \pi(p,w)}{\partial w_i} = -x_i(p,w)$

or $x_i(p,w) = -\dfrac{\partial \pi(p,w)}{\partial w_i}$

So $\dfrac{\partial x_i(p,w)}{\partial w_i} = -\dfrac{\partial^2 \pi(p,w)}{\partial w_i^2}$

We know from the properties of the profit function that it is convex, so $\dfrac{\partial^2 \pi(p,w)}{\partial w_i^2} \geq 0$

Therefore $\dfrac{\partial x_i(p,w)}{\partial w_i} \leq 0$ (unconditional input demand curves can't slope up)

2(c) Without differentiability, we need the algebraic method. Suppose we have data $(y^1, p^1; \ldots y^T, p^T)$ where each y^t is an n-dimensional production plan with $y_i^t \leq 0$ if i is an input and $y_i^t \geq 0$ if i is an output. The price vectors are positive.

From WAPM we know $p^t y^t \geq p^t y^s$ and $p^s y^s \geq p^s y^t$ $\Big\}$ all s, t

Define $\Delta y = y^t - y^s$.

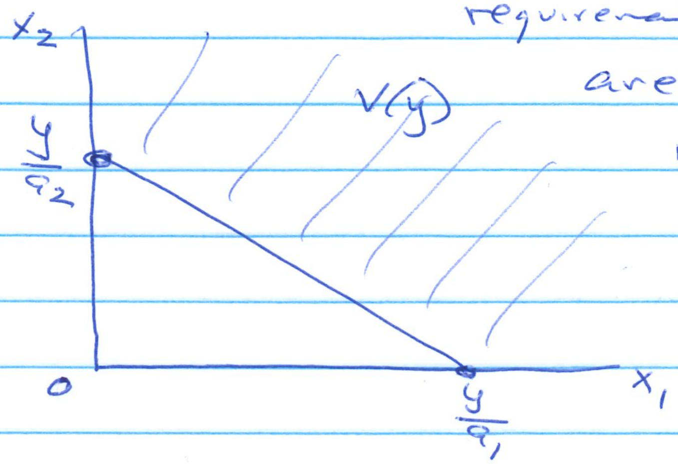
The inequalities give $p^t \Delta y \geq 0$ and $-p^s \Delta y \geq 0$

Summing these gives $\Delta p \Delta y \geq 0$ where $\Delta p = p^t - p^s$

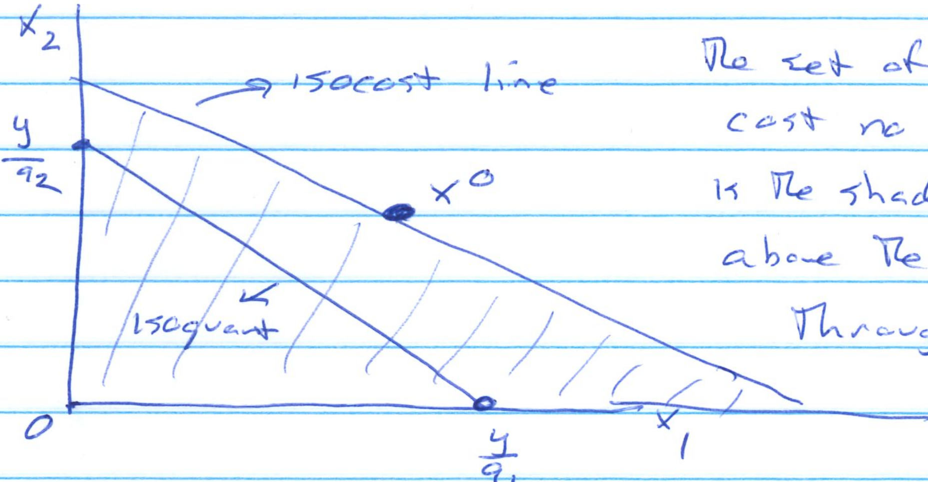
Now suppose y_i is an input. Set $\Delta p_i > 0$ with $\Delta p_j = 0$ for all $j \neq i$. We must have $\Delta y_i \geq 0$. This implies that when the price of an input rises, the quantity of the input cannot become <u>more</u> negative, so in absolute value the quantity cannot increase. Therefore input demand functions cannot slope up.

3(a) Write profit as $pf(x) - wx = pax - wx$
$= (pa - w)x = \sum_{i=1}^{n} (pa_i - w_i)x_i$. Because there is no upper bound on any x_i, if $pa_i - w_i > 0$ for any i, the problem does not have a solution. Now suppose $pa_i - w_i \leq 0$ for all i. In this case the maximum profit is zero and this can be achieved by setting $x_i = 0$ for all i. However, if $pa_i - w_i = 0$ for any i, the solution is not unique because any $x_i \geq 0$ is optimal. Therefore uniqueness requires $pa_i - w_i < 0$ for all i. In this case the unique solution is $x_i = 0$ for all i.

3(b) Consider the problem min wx subject to $ax = y$. When $n = 2$, the isoquant for y is linear, and the input requirement set $V(y)$ is the shaded area.



We know $V(y)$ is non-empty so choose some point $x^0 \in V(y)$ and consider the isocost line passing through it (determined by w)



The set of all points with cost no higher than x^0 is the shaded area. Nothing above the isocost line through x^0 can be optimal.

Now consider the intersection of $V(y)$ from the top graph with the shaded triangle from the bottom graph. The intersection is non-empty because x^0 is in both sets and it is bounded because the triangle is bounded. Both sets are closed, so the intersection is closed and bounded, therefore compact. The function wx we are minimizing is a continuous function of x. Therefore wx achieves a minimum at some x^* in the intersection set due to the Weierstrasse Theorem. No point outside the intersection can be a solution (it is either infeasible or has a cost greater than wx^0). So the problem has a solution.

3(c) Think of each i as a separate production technique. Let $y_i = a_i x_i$ be the output produced using technique i, so $y = \sum_{i=1}^{n} y_i$. The firm's expenditure for technique i is $w_i x_i = \frac{w_i y_i}{a_i}$. Therefore its total expenditure is

$$\sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} \frac{w_i y_i}{a_i}$$

The constraint is $y = \sum_{i=1}^{n} y_i$ so we can think of the firm's cost min problem as finding values for y_i that add up to y and give the lowest possible expenditure. Clearly if $\frac{w_i}{a_i} < \frac{w_j}{a_j}$ the firm can lower expenditure by reducing y_j a bit and increasing y_i by the same amount, so the constraint $\sum_i y_i = y$ is still satisfied. This logic shows that the firm will never use any method j if there is some i with $\frac{w_i}{a_i} < \frac{w_j}{a_j}$. If there is a unique i for which $\frac{w_i}{a_i}$ is at a minimum, the firm only uses that method, and

$c(w,y) = y \min_i \left\{ \frac{w_i}{a_i} \right\}$ (the entire y is produced using method i).

More generally, the firm could use some combination of techniques as long as they all have the same (smallest) ratio $\frac{w_i}{a_i}$.

4(a) When $a<0$, we have $f(0)=0$, $f'(x) = -ae^{ax} > 0$ and $f''(x) = -a^2 e^{ax} < 0$, so the marginal product is positive and decreasing in x. This makes sense.

When $a=0$ we have zero output for all input levels.

When $a>0$ we have $f'(x) < 0$ and $f''(x) < 0$ so output is negative for $x > 0$. Neither of these cases makes economic sense.

4(b) $p\left[1-e^{ax}\right] - wx + \mu x$ where μ is K-T multiplier.

FOC: $-pae^{ax} - w + \mu = 0$, $\mu \geq 0$, $x \geq 0$, $\mu x = 0$.

$\boxed{\text{If } x = 0}$ we have $-pa - w + \mu = 0$ where $\mu \geq 0$

implies $\mu = pa + w \geq 0 \Rightarrow \boxed{a \geq -\frac{w}{p}}$

$\boxed{\text{If } x > 0}$ we have $\mu = 0$ So $-pae^{ax} - w = 0$

$\Rightarrow e^{ax} = \frac{w}{-pa} \Rightarrow ax = \ln\left(\frac{w}{-pa}\right)$

$\Rightarrow \boxed{x(p, w) = \frac{1}{a}\ln\left(\frac{w}{-pa}\right)}$

Note that $x(p, w) > 0$ because

$\frac{1}{a} < 0$ but $\boxed{a < -\frac{w}{p}} \Rightarrow 1 > \frac{-w}{pa} \Rightarrow \ln\left(\frac{w}{-pa}\right) < 0$.

(c) If the firm wants output y, it must use the value of x

such that $y = 1 - e^{ax} \Rightarrow e^{ax} = 1 - y \Rightarrow ax = \ln(1-y)$

<u>Note:</u> it is impossible for $\Rightarrow x = \frac{1}{a}\ln(1-y)$.

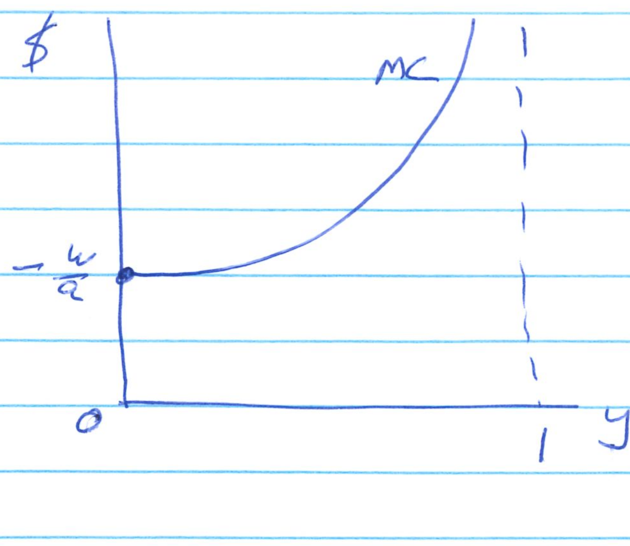
this firm to produce $y \geq 1$, so we limit attention to

cases where $0 \leq y < 1$.

The cost function is $c(w, y) = \frac{w}{a}\ln(1-y)$

Marginal cost is $\frac{\partial c(w,y)}{\partial y} = \frac{w}{a}\frac{1}{(1-y)}(-1) = -\frac{w}{a(1-y)}$

So the graph is



Note that $a < 0$

implies the vertical

intercept is positive;

$-\frac{w}{a} > 0$.

5(a). We want to show that $\pi(tp) = t\,\pi(p)$ for any $t > 0$.

Consider some arbitrary price vector p^*.

Suppose y^* is optimal for p^*. This means that

$$p^* y^* \geq p^* y \quad \text{for all } y \in Y.$$

Therefore $(tp^*)y^* \geq (tp^*)y$ for all $y \in Y$.

This shows that y^* is also optimal for (tp^*).

So $\pi(tp^*) = (tp^*)y^* = t(p^* y^*) = t\,\pi(p^*)$.

But p^* was arbitrary so this is true for all p.

Hence $\pi(tp) = t\,\pi(p)$ for any $t > 0$.

(b) Using the definition of a convex function, we need to show $\pi(p'') \leq t\,\pi(p) + (1-t)\,\pi(p')$ where $p'' = tp + (1-t)p'$ and $0 \leq t \leq 1$.

Let y'' be optimal for p''. Then

$$\pi(p'') = p'' y'' = [tp + (1-t)p']\,y''$$
$$= tpy'' + (1-t)p'y''$$
$$\leq t\,\pi(p) + (1-t)\,\pi(p')$$

because $\left.\begin{array}{l} \pi(p) \geq py'' \\ \pi(p') \geq p'y'' \end{array}\right\}$ in each case the max profit is at least as large as the profit from y''!

(c) Use the Lagrangean $L = wx - \lambda[f(x) - y]$ where $f(x) = x_1^{\alpha} x_2^{\beta}$ with $\alpha > 0$, $\beta > 0$.

The FOC $\Rightarrow$

$$w_1 = \lambda f_1(x) = \lambda \alpha x_1^{\alpha-1} x_2^{\beta}$$
$$w_2 = \lambda f_2(x) = \lambda \beta x_1^{\alpha} x_2^{\beta-1}$$

5 (c) cont.

Now multiply the first equation by x_1 and the second equation by x_2. This gives

$$w_1 x_1 = \lambda \alpha x_1^{\alpha} x_2^{\beta}$$
$$w_2 x_2 = \lambda \beta x_1^{\alpha} x_2^{\beta}$$

So $$z = \frac{w_1 x_1}{w_1 x_1 + w_2 x_2} = \frac{\lambda \alpha f(x)}{\lambda \alpha f(x) + \lambda \beta f(x)}$$
$$= \frac{\alpha}{\alpha + \beta}.$$

This shows that the cost share for input 1 does not depend on w or y.

Interpretation: The Cobb-Douglas function is a special case of the CES function where the elasticity of substitution is always equal to one. As we showed in class, $\sigma = 1$ implies that price effects and quantity effects cancel out, so when input prices change there is no effect on the distribution of expenditures between the two inputs.